ON STAGNANT FLOW REGIONS OF A VISCOUS-PLASTIC MEDIUM IN PIPES

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In paper [1] the problem of established flow of an incompressible viscous-plastic medium in pipes with arbitrary cross section was examined; theorems of existence and uniqueness of the solution were proven; a qualitative investigation of the flow character was carried out. Necessary and sufficient conditions of existence of motion with velocity different from zero were established. The existence of at least one rigid nucleus within the domain was proven. A sufficiently large class of cross sections was isolated for which the nucleus is unique,

In this work two questions are examined which were not touched upon in [1]:

firstly, the existence of stagnant regions in flow through pipes;

secondly, the mathematical side of the problem connected with the non-differentiability of the functional under examination.

The answer to the second question permits the conclusion that in the case under consideration, the equation of Euler remains valid only in regions where the solution has a velocity field gradient different from zero. In regions however where the solution has a constant value, Euler's equation is replaced by some natural geometric conditions amenable to clear physical interpretation. So, for example, such a condition for a rigid nucleus turns out to be the dynamic condition of its motion as a solid body. It should be noted that such conditions were earlier introduced into the problem as supplementary assumptions.

An analogous situation exists also for stagnant regions. In Section 1 of this paper necessary and sufficient conditions are formulated which are satisfied by the function which minimizes the initial functional. It is shown that the boundaries of the stagnant regions are always curved towards the stagnant zone and at each point have a curvature no less than τ_0/c , while the boundaries of nuclei at points of bulging have, conversely, a curvature no greater than τ_0/c.

In Section 2 it is shown that certain exact solutions for problems of motion of a viscous-plastic medium in pipes actually minimize the corresponding functionals. The possibility of existence of stagnant zones is proven depending on geometrical peculiarities of the boundary (corner points, regions with reduced width)

Results from Section 3 of [1] are frequently used in this paper. For this reason all notations adopted there are retained; just as in the paper [1] all cumbersome proofs are placed in an appendix at the end of the paper.

1. Criterion for selection of true motion. We shall examine the functional

$$J(u) = \int_\omega \left\{ \frac{\mu}{2} (\nabla u)^2 + \tau_0 \,|\nabla u| - cu \right\} d\omega \tag{1.1}$$

defined for functions $u(x, y)$ which are continuous together with the first partial derivatives within the confines of the bounded domain ω and which satisfy the following boundary conditions on the boundary Γ of the domain:

$$u \,|\, \Gamma = \varphi(x,y) \tag{1.2}$$

In [1] it was shown that the function which describes the real motion of the viscous-plastic medium in the pipe with an arbitrary cross section, minimizes the functional (1.1).

The purpose of this section is to find effective conditions which permit a check that the specified sufficiently smooth function $u_0(x, y)$ subject to condition (1.2) minimizes the functional (1.1).

Let us assume that the point set of the domain ω, where $|\nabla u_0| = 0$, represents the totality of closed nonintersecting domains $A_1, \ldots, A_s$ and $B_1, \ldots, B_p$ where all A_i are located strictly within ω, while each B_i has at least one common point with Γ. The boundary of the domain A_i is designated by a_i, the boundary of the domain B_i is designated by b_i. With respect to $u_0(x, y)$ it is also assumed that it achieves its local maximum in each A_i and that in the domain Ω, which is the part of the domain ω where $|\nabla u_0| > 0$, it is continuous together with its derivatives through, inclusively, second order. In the following text we shall refer to domain A_i as nuclei of flow and to domains B_i as stagnant zones.

Necessary and sufficient conditions which must be satisfied by the function minimizing the functional (1.1) can be formulated in the form of the following theorem.

T h e o r e m * 1.1 (criterion) (*). For the function $u_0(x, y)$ to minimize functional (1.1) it is necessary and sufficient that:

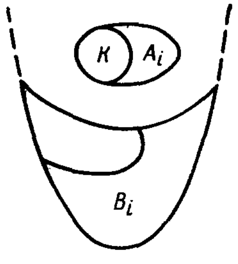

Fig. 1

1. In the region Ω the function $u_0(x, y)$ satisfies Equation

$$\mu \Delta u_0 + \tau_0 \operatorname{div} [\nabla u_0 / |\nabla u_0|] + c = 0$$

2. In each domain B_j, for any contour K which is located in B_j and which is the boundary for subdomain K^* of domain B_j, the following inequality holds (Fig.1) (**)

$$\tau_0 \operatorname{mes} L \geqslant c \operatorname{mes} K^* + \tau_0 \operatorname{mes} \gamma \quad (K = L + \gamma)$$

where γ is the part of contour K which coincides

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*) Proofs of Theorems and Lemmas designated by an asterisk are given in the appendix.

**) By mes L, mes γ and mes K^* the corresponding length of lines L and γ and the area of domain K are designated.

with $b_j \setminus \Gamma$ (*).

3. In each domain A_i the following relationships hold:

$$\text{a) } \tau_0 \operatorname{mes} a_i = c \operatorname{mes} A_i, \quad \text{(b) } \tau_0 \operatorname{mes} K \geqslant c \operatorname{mes} K^*$$

where K is an arbitrary contour lying in the domain A_i and forming the boundary for sub-domain K^* of the domain A_i.

While conditions 2 and 3 of the criterion have a purely geometrical character they are difficult to verify by virtue of the arbitrariness of contour K which enters in. Lemmas 1.1 and 1.2 make the practical utilization of the criterion substantially easier. These Lemmas isolate a comparatively narrow class of contours on which it is appropriate to check conditions 2 and 3 of criterion.

We shall examine domain D with boundary d. Let K be a contour located within the confines of region D and forming the boundary of sub-domain K^* of domain D.

L e m m a * 1.1 . Functional $M(K) = \tau_0 \operatorname{mes} K - c \operatorname{mes} K^*$ achieves its minimum on contour K' with the following properties.

1. In internal points D the contour K' coincides with another periphery with radius τ_0/c.

2. Contour K' can approach boundary d at a nonzero angle only at points where the boundary d is not smooth.

Let the boundary d be representable in the form $d = \gamma + L$ where γ is the totality of a finite number of smooth curves. Then the contour K examined above permits the representation $K = T + \tau$ where τ is part of γ.

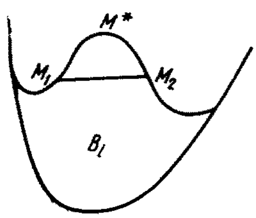

Fig. 2

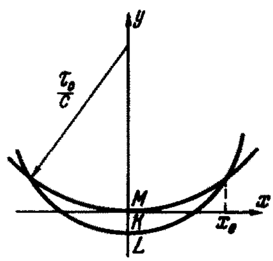

Fig. 3

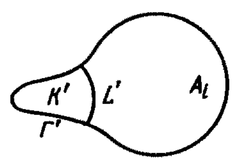

Fig. 4

L e m m a 1.2 . Functional $N(K) = \tau_0 \operatorname{mes} T - \tau_0 \operatorname{mes} \tau - c \operatorname{mes} K^*$ attains its minimum on contour K' which has the properties 1 and 2 of Lemma 1.1.

Proof of Lemma 1.2 is a word for word repetition of proof of Lemma 1.1.

Conditions 2 and 3 of the criterion permit to draw certain conclusions with regard to geometrical peculiarities of boundaries of stagnant zones and

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*) $b_i \setminus \Gamma$ designates the set of points of curve b_i which do not lie on Γ.

nuclei of flow. First of all, it is completely obvious that none of the domains A_i or B_i can contain a circle with a radius greater than $2\tau_0/c$. Secondly, it is easy to see that $b_i \setminus \Gamma$ is concave with respect to region B_i.

In fact, let us assume the opposite. Let us examine the contour $K = \smile M_1M^*M_2 + [M_1, M_2]$ (see Fig.2). It is apparent that in this case mes $\smile M_1M^*M_2 >$ mes$[M_1,M_2]$, contradicts condition 2 of criterion. Less apparent is the following property of the curve $b_i \setminus \Gamma$.

T h e o r e m 1.2 . If $b_i \setminus \Gamma$ is a curve with continuously varying curvature $\varkappa$, then $|\varkappa| \leq c/\tau_0$.

P r o o f . Let us assume the opposite. Then a point N exists on $b_i \setminus \Gamma$, where $|\varkappa| > c/\tau_0$. Let us examine the vicinity of point N and introduce in this vicinity new coordinates orienting the axis O_x along the tangent to the curve and the axis O_y along the normal. The origin of coordinates is selected at the point N. The curve $b_i \setminus \Gamma$ in the vicinity of point N can be represented in the form $y = ax^2 + O(x^3)$, $a < c/2\tau_0$. A periphery is drawn with a radius τ_0/c as is shown in Fig.3. We note that such construction is possible for sufficiently small x_0, namely, because of $a < c/2\tau_0$.

As contour K we select a contour consisting of an arc of periphery L and an arc of curve τ. By K^* we designate a domain bounded by contour K. It is easy to find that

$$\text{mes } K^* = -{}^4/_3 a x_0^3 - (\tau_0/c)\, x_0 + c x_0^3/2\tau_0 + (\tau_0/c)^2 \sin^{-1}(c x_0/\tau_0) + O(x_0^4)$$

$$\text{mes } \tau = 2x_0 + {}^4/_3 a^2 x_0^3 + O(x_0^4)$$

$$\text{mes } L = (2\tau_0/c) \sin^{-1}(c x_0/\tau_0)$$

It follows from condition 2 of criterion that

$$\tau_0 \text{ mes } L \geqslant \tau_0 \text{ mes } \tau + c \text{ mes } K^* \tag{1.3}$$

Substituting into this inequality values found for mes L, mes τ and mes K^*, we obtain $O \geqslant (2a\tau_0 - c)^2 x_0^3 + O(x_0^4)$, which is impossible for sufficiently small x_0. Theorem 1.2 is proven.

The following Theorem is proven quite analogously.

T h e o r e m 1.3 . If the boundary a_i of the nucleus of flow A_i at the point N is convex and the curvature $\varkappa$ of the boundary is continuous in N, then in N

$$|\varkappa| > c/\tau_0$$

In the proof of Theorem 1.3 instead of condition (1.3) it is appropriate to make use of the following inequalities which result from relationships (a) and (b) of point 3 of the criterion

$$\tau_0 \text{ mes } \Gamma' \leqslant \tau_0 \text{ mes } L' + c \text{ mes } K'^*$$

Notations Γ', L' and K'^* are indicated in Fig.4.

Conditions 2 and 3 of Theorem 1.1 have a clear physical significance. If conditions for motion of the nucleus are set up as of a solid body without acceleration, they will have the form

$$\tau_0 \,\mathrm{mes}\, a_i = c \,\mathrm{mes}\, A_i$$

It is clear that if conditions of equilibrium of all forces acting on the nucleus are fulfilled for the whole nucleus in its entirety, then they must be fulfilled a fortiori for any of its parts. An analogous situation exists also for stagnant zones.

In this manner conditions 2 and 3 of Theorem 1.1 represent dynamic conditions for motion of nuclei and equilibria of stagnant zones.

2. Verification of known exact solutions. The criterion formulated in Section 1 for the selection of real motion of a viscous-plastic medium in pipes from all kinematically possible motions permits verification of known points of solution [2 to 4].

1. M o t i o n i n a c i r c u l a r p i p e [2]. In this case the exact solution has the following form (Fig.5):

$$u_0 = \frac{\tau_0}{\mu} \ln \frac{r}{R} + \frac{c}{4\mu}(R^2 - r^2) \quad \text{for} \quad R_1 \leqslant r \leqslant R$$
$$u_0 = \frac{\tau_0}{\mu} \ln \frac{R_1}{R} + \frac{c}{4\mu}(R^2 - R_1^2) \quad \text{for} \quad 0 \leqslant r \leqslant R_1 \qquad \left(R_1 = \frac{2\tau_0}{c}\right) \qquad (2.1)$$

Condition 1 of Theorem 1.1 is verified by direct substitution of $u_0(r)$ into the differential equation. Since stagnant zones are absent, condition 2 drops out. Consequently, it is necessary to check only condition 3 of Theorem 1.1. In the case under examination the nucleus is unique and its boundary Γ is a periphery of radius R_1. Lemma 1.1 permits the assertion that condition 3 of criterion 1.1 must be checked on two contours. One of these is the periphery of radius τ_0/c and the other is the periphery of radius $2\tau_0/c$. In both cases condition 3 of Theorem 1.1 is satisfied. This also proves that function $u_0(r)$ minimizes functional 1.1.

2. L o n g i t u d i n a l m o t i o n i n a n a n n u l a r g a p [3]. The exact solution is given by the following equation (Fig.6)

$$u_0 = \frac{\tau_0}{\mu}(R_1 - r) + \left[\frac{\tau_0 R_2}{\mu} + \frac{CR_2^2}{2\mu}\right] \ln \frac{r}{R_1} + \frac{c}{4\mu}(R_1^2 - {}^2) \qquad \text{for} \quad R_1 \leqslant r \leqslant R_2$$

$$u_0 = \frac{\tau_0}{\mu}(r - R_4) + \left[-\frac{\tau_0 R_3}{\mu} + \frac{cR_3^2}{2\mu}\right] \ln \frac{r}{R_4} + \frac{c}{4\mu}(R_4^2 - r^2) \qquad \text{for} \quad R_3 \leqslant r \leqslant R_4$$

$$u_0 = \frac{\tau_0}{\mu}(R_1 - R_2) + \left[\frac{\tau_0 R_2}{\mu} + \frac{cR_2^2}{2\mu}\right] \ln \frac{R_2}{R_1} + \frac{c}{4\mu}(R_1^2 - R_2^2) \qquad \text{for} \quad R_2 \leqslant r \leqslant R_3$$

$$\frac{\tau_0}{\mu}(R_1 - R_2) + \left[\frac{\tau_0 R_2}{\mu} + \frac{CR_2^2}{3\mu}\right] \ln \frac{R_2}{R_1} + \frac{c}{4\mu}(R_1^2 - R_2^2) = \frac{\tau_0}{\mu}(R_3 - R_4) +$$
$$+ \left[-\frac{\tau_0 R_3}{\mu} + \frac{cR_3^3}{2\mu}\right] \ln \frac{R_3}{R_4} + \frac{C}{4\mu}(R_4^2 - R_3^2) \qquad R_3 - R_2 = \frac{2\tau_0}{c} \qquad (2.2)$$

Condition 1 is checked exactly the same way by direct substitution into Euler's equation. Stagnant zones are absent and condition 2 of Theorem 1.1 is eliminated. Condition 3 of criterion 1.1 must be verified again on two contours. One contour is a periphery with a radius τ_0/c which is inscribed with tangency into the nucleus itself. The second contour consisting of two parts is the boundary of the nucleus itself. In both cases condition 3 is fulfilled, this proves that function u_0 (2.2) minimizes (1.1)

3. F l o w i n n o n c i r c u l a r p i p e s [4]. The solution for a pipe with noncircular cross section obtained in [4] minimizes functional (1.1). This fact is obtained fairly simply by utilizing Lemma 1.1, but requires cumbersome computations which are omitted for the sake of

brevity. Let us approach the examination of stagnant zones. In [4] the exact solution u_0 is constructed in an angular domain $(\alpha > \frac{1}{4}\pi)$. In this case the function u_0 becomes zero (Fig.7) somewhere in the vicinity of the tip of the angle bounded by sides OA, OB and the curve γ. We draw a periphery with radius OR, with the center at the point O. The function u_0 becomes zero on lines OR_1 and OR_2 and takes the value $\varphi(x, y) > 0$ on the arc of periphery R_1MR_2.

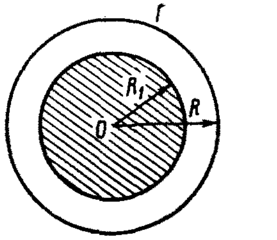


Fig. 5

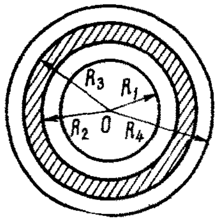


Fig. 6

We shall demonstrate that among all functions which become zero on radii OR_1 and OR_2 and are equal to $\varphi(x, y)$ on the arc R_1MR_2, the function u_0 gives the smallest value to the functional (1.1). To convince oneself of this it is sufficient to verify conditions 1 and 2 of the criterion. There are no nuclei of flow here, therefore condition 3 drops out. Condition 1 of criterion is easily verified by direct substitution of u_0 into the corresponding differential equation. We shall check condition 2 of criterion. Since the radius of curvature R of curve γ is equal to $(\tau_0/c)\{1 + [4a(B + \cos^2\varphi)]^{-1}\}$ (for notations see [4]), in the domain of the stagnant zone it is impossible to draw a periphery tangent to the boundaries of the stagnant zone.

Utilizing the statement of Lemma 1.2 it is found that condition 2 of criterion must be checked only on two contours. The first contour K_1 represents the boundary of the stagnant zone, the second contour K_2 is degenerate and represents the arc γ which is passed twice. We note that the first contour is not external for functional $N(K)$ (see Lemma 1.2) since for contour $K_3 = \gamma + APQB$ (Fig.7)

$$N(K_1) - N(K_3) = \rho\left(\frac{2\tau_0(1-\sin\alpha)}{\cos\alpha}\right) - c\rho\cot\alpha\Big), \quad \rho \leqslant \frac{\tau_0(1-\sin\alpha)}{c\sin\alpha} \tag{2.3}$$

and $N(K_1) \geq N(K_3)$. From this it follows that $\inf N(K)$ is achieved on a degenerate contour K_2 and $\inf N(K) = 0$. Condition 2 of criterion has been verified. We note that solution u_0 in this case minimizes functional (1.1) not only in the sector under examination, but also in the domain represented in Fig.8a) if only the curve L does not touch the boundary γ.

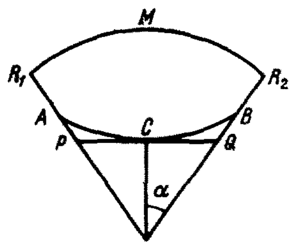
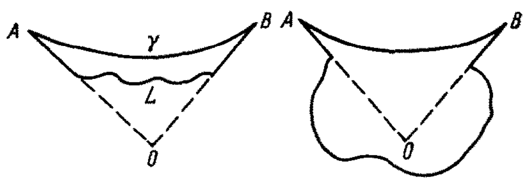


Fig. 7



a) b)

Fig. 8

In this manner the outer boundary of the stagnant zone can be deformed in an arbitrary manner within the sector without touching the boundary γ, while the solution u_0 in the flow domain will remain unchanging. We can also examine the growth of stagnant zone (Fig.8, b) which preserves solution u_0 unchanged in the domain of the flow. The boundary L in this case is not arbitrary, but for example such that in the domain bounded by curves γ and L (Fig.8, b) it is not possible to draw a periphery with a radius τ_0/c which touches the boundaries. For such choice of L condition 2 of criterion is verified in an obvious manner with utilization of Lemma 1.2. This indicates that the region between γ and L (Fig.8, b) is a stagnant zone of flow.

The solution found in the angular domain permits to find the exact solution u_0 in the domain represented in Fig.9. Solution u_0 becomes zero on segment R_1T, $R_1'T$, R_2S and $R_2'S$ and it becomes $\varphi(x, y)$ on arcs of peripheries R_1R_2 and $R_1'R_2'$. The domain represented ib Fig.9 is obtained by superposition of sectors (Fig.7) on one another. It in appropriate to note that superposition of sectors must not be very large if it is required to keep the flow domain unchanged. For example, if the sectors are superimposed such that the curves γ in the upper and lower sector touch (Fig.10), then in this case the distribution changes in the flow dimain because domain K bounded by the broken line ATA' and two segments (AC, AC') of curve γ does not satisfy condition 2 of the criterion

$$\tau_0 \operatorname{mes}(ATA') - \tau_0 \operatorname{mes}(ACA') - c \operatorname{mes} K < 0$$

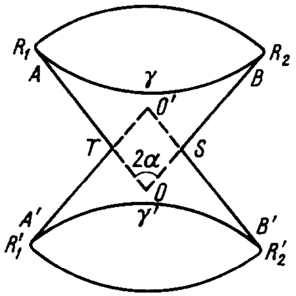


Fig. 9

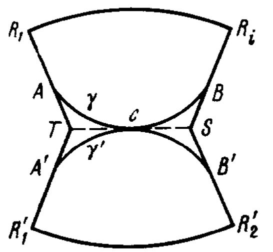


Fig. 10

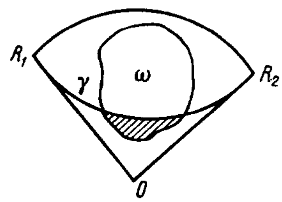


Fig. 11

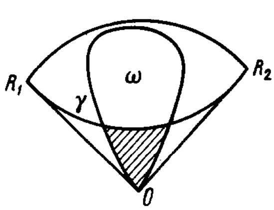


Fig. 12

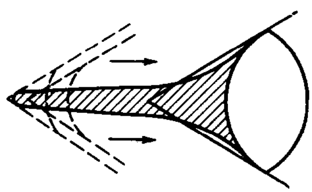


Fig. 13

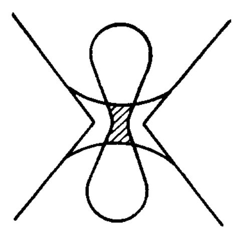


Fig. 14

We denote the quantity OT in Fig.10 by λ. Then it follows from Lemma 1.2 and relationship (2.3) that the region of flow will remain invariant if

$$2\rho\tau_0 \frac{1-\sin\alpha}{\cos\alpha} - c\rho^2 \tan\alpha - 2\lambda\tau_0 + \lambda^2 c \sin\alpha > 0, \qquad 0 < \lambda < \lambda_1$$

In the following only such superpositions of sectors are examined for which the stagnant zone takes up the domain between curves γ in the upper and lower sectors (Fig.10). We shall demonstrate now that for steady flow of a viscous-plastic medium in cylindrical tubes with arbitrary cross section, stationary zones can exist, i.e. zones adjacent to tube walls where the velocity is equal to zero. This fact will follow from the majorizing principle presented in [1]. Let ω_1 and ω_2 be two plane domains and domain ω_1 be part of ω_2. Let u_1 and u_2 be functions minimizing functional (1.1) in the domains ω_1 and ω_2, respectively. Then $0 \le u_1 \le u_2$ in ω_1. We shall assume that the bounded domain ω is located within the obtuse angle. We shall examine a sector OR_1R_2 of sufficiently large radius so that ω is located within the sector (Fig.11).

Let us examine function u_0 constructed in [4] for the angle. From the majorization principle it follows that $u \le u_0$, where u is a function minimizing functional (1.1) in ω and becoming zero at the boundary Γ of the domain ω. However, since u_0 is equal to zero below the curve γ (Fig.11), u also becomes zero in ω everywhere below the curve γ if curve γ crosses the domain ω. Thus, in this case in the domain ω there exists a stagnant zone taking up at least the domain hatched in Fig.11.

It follows from presented arguments that if ω has a corner point and can be located in the obtuse angle with apex in the corner point of ω (for example ω is convex), then a stagnant zone exists in the domain ω (Fig.12). After the existence of the stagnant zone has been established in the domain ω, it is natural to attempt to find the greatest possible subdomain of domain ω which will fit into the stagnant zone. In a number of cases this can be achieved by transposing the sector so that its apex moves in the stagnant zone of domain ω. This motion of the sector over the stagnant zone is represented in Fig.13.

We shall examine another interesting case of stagnant zones having the character of cross members separating two or several regions of flow (Fig. 14). The existence of such stagnant zones follows from the exact solution constructed in the domain $R_1TR_1'R_2'SR_2$ presented in Fig.9 and the majorizing principle. Additional examination of dimensions of the stagnant zone can be carried out by the method presented above. In conclusion we note a simple sufficient condition for the absence of a stagnant zone in the vicinity of a boundary point. If the boundary point can be touched by a circle of radius $2\tau_0/c$ which is located completely in the domain ω, then in the vicinity of this point a stagnant zone is not present. This sufficient condition is a trivial consequence of the majorizing principle and the exact solution examined in Section 2, point 1.

Appendix. At first we shall establish some auxiliary statements.

D e f i n i t i o n . A function v_0 which satisfies (1.2) gives weak minimum of functional (1.1) if for any smooth function h, $h|_\Gamma = 0$ there is a value λ_0 such that all λ, $|\lambda| \le \lambda_0$

$$J(v_0 + \lambda h) \geqslant J(v_0) \tag{A.1}$$

L e m m a A.1 . If v_0 gives a weak minimum to functional (1.1), then $v_0 = u_0$, where u_0 is a function giving an absolute minimum to functional (1.1).

P r o o f . By virtue of convexity of functional (1.1) we have the inequality

$$J(v_0 + \lambda(v_0 - u_0)) \leqslant J(v_0) + \lambda[J(u_0) - J(v_0)], \quad 0 \leqslant \lambda \leqslant 1 \tag{A.2}$$

Let us select a smooth function h, $h|_\Gamma = 0$ and such that

$$\int_\omega \{|\nabla(h - (v_0 - u_0))|^2 + |h - (v_0 - u_0)|^2 d\omega < \delta \tag{A.3}$$

Then

$$|J(v_0) - J(v_0 + \lambda(h - (v_0 - u_0)))| = \Big|\int_\omega \{\lambda\mu\nabla(h - (v_0 - u_0))\nabla v_0 +$$

$$+ \lambda^2\frac{\mu}{2}|\nabla(h - (v_0 - u_0))|^2 + \tau_0|\nabla v_0| - \tau_0|\nabla(v_0 + \lambda(h - (v_0 - u_0)))| -$$

$$- c\lambda(h - (v_0 - u_0))\} d\omega|$$

Since

$$\Big|\int_\omega \{|\nabla v_0| - |\nabla(v_0 + \lambda(h - (v_0 - u_0)))|\}|d\omega| \leqslant |\lambda| \int_\omega |\nabla(h - (v_0 - u_0))| d\omega$$

it therefore follows from (A.3) that

$$|J(v_0) - J(v_0 + \lambda(h - (v_0 - u_0)))| \leqslant |\lambda|\delta K \tag{A.4}$$

where K is independent of λ and δ. We have further from (A.2) that

$$J(v_0) \geqslant \lambda[J(v_0) - J(u_0)] + J(v_0 + \lambda(v_0 - u_0)) \geqslant \lambda[J(v_0) - J(u_0)] +$$
$$+ 2J(v_0 + {}^1/_2\lambda h) - J(v_0 + \lambda(h - (v_0 - u_0))) \geqslant \lambda[J(v_0) - J(u_0)] +$$
$$+ 2J(v_0 + {}^1/_2\lambda h) - J(v_0) - |J(v_0) - J(v_0 + \lambda(h - (v_0 - u_0)))| \geqslant$$
$$\geqslant \lambda[J(v_0) + J(u_0)] + 2J(v_0 + {}^1/_2\lambda h) - J(v_0) - \lambda K\delta$$

Consequently

$$J(v_0) \geqslant (1/_2\lambda)\,[J(v_0) - J(u_0)] + J(v_0 + 1/_2\lambda h) - 1/_2\lambda K\delta$$

If $J(v_0) > J(u_0)$ then δ can be selected so small that we will have $J(v_0) - J(u_0) > 2K\delta$ then for all $\lambda (0 \le \lambda \le 1)$

$$J(v_0) \geqslant J(v_0 + 1/_2\lambda h) \tag{A.5}$$

Inequality (A.5) contradicts (A.1), consequently $J(v_0) = J(u_0)$. From the theorem of uniqueness [1] it follows that $v_0 = u_0$. The Lemma has been proven.

L e m m a A2 . If functional (1.1) has a critical point v_0, then $v_0 = u_0$ where u_0 is a function which minimizes functional (1.1).

P r o o f . Let v_0 be a critical point, then

$$\lim_{\lambda\to+0} \frac{J(v_0 + \lambda(v_0 - u_0)) - J(v_0)}{\lambda} = 0 \tag{A.6}$$

However, $J(v_0 + \lambda(v_0 - u_0)) = J(\lambda u_0 + (1-\lambda)v_0) \leqslant \lambda J(u_0) + (1-\lambda)J(v_0)$,

i.e. $$J(v_0 + \lambda(v_0 - u_0)) - J(v_0) \leqslant \lambda\,[J(u_0) - J(v_0)] < 0$$

Consequently

$$\lim_{\lambda\to+0} \frac{J(v_0 + \lambda(u_0 - v_0)) - J(v_0)}{\lambda} \leqslant J(u_0) - J(v_0) < 0$$

The last inequality contradicts (1.6), if $u_0 \neq v_0$. Lemma A2 is proven.

L e m m a A3 (*) . For the inequality

$$\tau_0 \int_\omega |\nabla h|\,d\omega + \tau_0 \int_\Gamma h\,ds \geqslant c \int_\omega h\,d\omega \tag{A.7}$$

to be applicable for any smooth $h(x, y)$, it is necessary and sufficient

$$1^\circ \tau_0 \operatorname{mes}\Gamma = c \operatorname{mes}\omega, \quad 2^\circ \tau_0 \operatorname{mes}\Gamma' \geqslant c \operatorname{mes}\omega'$$

where ω' is an arbitrary sub-domain of domain ω and Γ' is the boundary of ω'.

P r o o f . N e c e s s i t y . Condition 1 follows from (A.7) if we write $h(x, y) = H$ is a constant. Let us examine an arbitrary sub-domain ω' of domain ω with the boundary Γ'. Let Γ' have a finite curvature in all points. Then in some neighborhood of this boundary $O_j(\Gamma')$ we can introduce a curvilinear system of coordinates using as one variable s the length of the arc along Γ' and as the other variable n the length of segment normal to Γ'. The boundary of $O_j(\Gamma')$ is made up of lines $n_j(s) = \pm\alpha_j$. Let us assume that $\alpha_j \to 0$ for $j \to \infty$ and that Γ' is imbedded in ω together with $O_j(\Gamma')$, starting with some j. Let us examine in ω the sequence of functions v_j equal to unity in $\omega' \setminus O_j(\Gamma')$ and to zero in $\omega \setminus (\omega' \cup O_j(\Gamma'))$. In $O_j(\Gamma')$ function v_j is a monotonous function of variable n. Then

$$\lim_{j\to\infty} \int_\omega |\nabla v_j|\,d\omega = \operatorname{mes}\Gamma', \qquad \lim_{s\to\infty} \int_\omega v_j\,d\omega = \operatorname{mes}\omega' \tag{A.8}$$

Condition 2 of Lemma follows from (A.7) and (A.8). It is easy to see that the condition of finite curvature and imbedding are unessential. Necessity of conditions 1 and 2 is proven.

S u f f i c i e n c y . It is sufficient to establish inequality (A.7) for arbitrary polynomials. Let $Q(x, y)$ be a polynomial; then it has only a finite number of level lines passing through singular points where $|(\nabla Q)| = 0$. Level lines Q passing through singular points will be called singular, other level lines will be called nonsingular.

*) Lemmas used in the proof and notations $A \setminus B$, $A \cup B$ and $U_p A_p$ used in examinations below [7] as usual denote the difference in sets A and B, the sum of sets A and B and the sum of the family of sets A_p, respectively.

Let us examine a nonsingular line of level L_ρ, the value on which is equal to ρ $(Q(L_\rho)=\rho)$. In some vicinity of L_ρ we can introduce a curvilinear system of coordinates s and n in the manner indicated above. In this vicinity we take the level line $L_{\rho+\Delta\rho}$, the equation for which is $n = n(s) > 0$. It will be assumed here that for fixed s the quantity $Q(n, s)$ is nondecreasing function of n for $Q \le n \le (s)$.

The set of points in the neighborhood under examination which belong to ω and are such that for them $0 \le n \le n(s)$, will be denoted by $\omega_{\rho,\rho+\Delta\rho}$. It is easy to see that the domain ω can be stratified into sub-domains of the type $\omega_{\rho,\rho+\Delta\rho}$, with accuracy to a polynomial of a degree as small as desired, i.e.

$$\omega_\varepsilon = \bigcup_\rho \omega_{\rho,\rho+\Delta\rho}, \quad \text{mes}\,[\omega \setminus \omega_\varepsilon] < \varepsilon$$

where $\varepsilon \to 0$ for $n(s) \to 0$. Through K_ρ we designate a closed contour coinciding with L_ρ, if L_ρ is an oval lying in ω and $K_\rho = L_\rho + \gamma_\rho$, and if L_ρ with its ends comes out on the boundary Γ of domain ω. Here γ_ρ is the part of boundary Γ which connects the ends of L_ρ, where $Q(\gamma_\rho) \geqslant \rho$. By ω_ρ we shall designate a sub-domain of ω which is bounded by contour K_ρ. Since

$$\tau_0\, \text{mes}\, K_\rho \geqslant c\, \text{mes}\, \omega_\rho$$

then

$$\tau_0\, [\text{mes}\, L_\rho + \text{mes}\, \gamma_\rho]\, [Q(L_{\rho+\Delta\rho}) - Q(L_\rho)] \geqslant c\, \text{mes}\, \omega_\rho\, [Q(L_{\Delta\rho+\rho}) - Q(L_\rho)] \qquad \text{(A.9)}$$

We note that

$$\tau_0\, \text{mes}\, L_\rho^{\cdot}\, [Q(L_{\rho+\Delta\rho}) - Q(L_\rho)] = \tau_0 \int\limits_{\omega_{\rho,\rho+\Delta\rho}} \frac{\partial Q}{\partial n}\, dn\, ds + O(n(s))\, n(s)$$

Summing (A.9) with respect to ρ we obtain

$$\sum_\rho \text{mes}\, \omega_\rho\, [Q(L_{\rho+\Delta\rho}) - Q(L_\rho)] = \int\limits_\omega \{Q(x, y) - \inf_\omega Q(x, y)\}\, d\omega + O(n(s))$$

$$\sum_\rho \text{mes}\, \gamma_\rho\, [Q(L_{\rho+\Delta\rho}) - Q(L_\rho)] = \int\limits_\Gamma \{Q(x, y) - \inf_\Gamma Q(x, y)\}\, ds + O(n(s))$$

$$\sum_\rho \text{mes}\, L_\rho\, [Q(L_{\rho+\Delta\rho}) - Q(L_\rho)] \leqslant \int\limits_\omega |\nabla Q|\, d\omega + O(n(s))$$

Consequently,

$$\tau_0 \int\limits_\omega |\nabla Q|\, d\omega + \tau_0 \int\limits_\Gamma Q\, ds \geqslant c \int\limits_\omega Q\, d\omega + \tau_0 \inf_\Gamma Q\, \text{mes}\, \Gamma - c \inf_\omega Q\, \text{mes}\, \omega$$

From condition 1 of Lemma

$$\tau_0 \inf_\Gamma Q\, \text{mes}\, \Gamma - c \inf_\omega Q\, \text{mes}\, \omega \geqslant \inf_\omega Q\, [\tau_0\, \text{mes}\, \Gamma - c\, \text{mes}\, \omega] = 0$$

In this fashion Lemma A.3 has been proven.

Let the domain ω be bounded by contour R and $R = \Gamma + \gamma$ where γ are some smooth curves consisting, generally speaking, of a finite number of connected components.

L e m m a A.4 . For the inequality

$$\tau_0 \int\limits_\omega |\nabla h|\, d\omega - \tau_0 \int\limits_\gamma h\, ds \geqslant c \int\limits_\omega h\, d\omega \qquad \text{(A.10)}$$

to be satisfied for all smooth h, which become zero on Γ, the fulfillment of the following condition is necessary and sufficient: for any closed contour $R' = \Gamma' + \gamma'$ lying in ω where γ' is part of γ the inequality

$$\tau_0\, \text{mes}\, \Gamma' - \tau_0\, \text{mes}\, \gamma' \geqslant c\ \text{mes}\, \omega' \qquad \text{(A.11)}$$

applies, where ω' is a sub-domain of ω bounded by the contour R'.

P r o o f . N e c e s s i t y . In analogy to Lemma A.3 we construct a step-wise function which is the limit of v_j, equal to 1 in the domain ω' and equal to zero outside the contour R'., Substituting this function into (A.10) we obtain condition (A.11). We shall demonstrate this. Let Γ' be a smooth curve; in its vicinity $O_j(\Gamma')$ we shall introduce curvilinear coordinates (s, n). Boundary of $O_j(\Gamma')$ are the lines $h = \pm \alpha_j$ and $\alpha_j \to 0$ for $j \to \infty$. Let us examine a sequence of functions $v_j(x, y)$ which are equal to unity in $\omega' \setminus Q_j(\Gamma')$ and to zero in $\omega \setminus [\omega' \bigcup O_j(\Gamma')]$. In $O_j(\Gamma')$ functions v_j are monotone functions of variable n. Then

$$\lim_{j\to\infty} \int_{\omega} |\nabla v_j| \, d\omega = \operatorname{mes} \Gamma', \quad \lim_{j\to\infty} \int_{\gamma} v_j \, ds - \operatorname{mes} \gamma', \quad \lim_{j\to\infty} \int_{\omega} v_j \, d\omega = \operatorname{mes} \omega' \quad (A.12)$$

Comparing (A.12) with (A.10) we obtain (A.11). The necessary condition has been proven.

S u f f i c i e n c y . We note [5] that it is sufficient to establish the inequality (A.10) on functions (*) from $W_p{}^1(\omega)$ $(p > 2)$ positive in ω and becoming zero on Γ. Let us approximate such a function by a polynomial $Q_n{}^1(x, y)$ in the metric $W_p{}^1(\omega)$. From theorems of imbedding [5] it follows that $Q_n{}^1$ converges to h uniformly, i.e.

$$|Q_n{}^1 - h| < \frac{1}{n}, \qquad \int_{\omega} |\nabla (Q_n{}^1 - h)| \, d\omega < \frac{1}{n} \qquad (A.13)$$

Let us examine the polynomial $Q_n = Q_n{}^1 - 1/n$. It is clear that $Q_n \leq 0$ on Γ.

From the polynomial Q_n we make the transition to function Q_n^*

$$Q_n^* = 0 \quad \text{for } Q_n \leqslant 0, \qquad Q_n^* = Q_n \quad \text{for } Q_n > 0$$

We shall demonstrate that for Q_n^* evaluations analogous to (A.13) are applicable. The set where $Q_n^* = 0$ is denoted by S_n. Then $|Q_n^* - h| < 2/n$ in $\omega \setminus S_n$. Since $|h| < 2/n$ in the domain S_n, then $|Q_n^* - h| < 2/n$ in S_n. We shall demonstrate (**) that $\operatorname{mes}\{S_n \bigcap \operatorname{supp} h\} \to 0$ for $n \to \infty$. Let us examine the set $\Lambda_n = \{(x, y) \,|\, |h| < 2/n\} \bigcap \operatorname{supp} h$. Then $\lim \Lambda_n = \Phi$, where Φ is an empty set and $\Lambda_n \supseteq \Lambda_{n+1}$. Consequently $\operatorname{mes} \Lambda_n \to 0$ for $n \to \infty$. But $\Lambda_n \supset S_n \bigcap \operatorname{supp} h$. In this manner $\operatorname{mes}\{S_n \bigcap \operatorname{supp} h\} \to 0$ for $n \to \infty$. Thus it follows from (A.13) directly that

$$|h - Q_n^*| \to 0, \qquad \int_{\omega} |\nabla (h - Q_n^*)| \, d\omega \to 0 \qquad \text{for } n \to \infty \qquad (A.14)$$

Utilizing relationship (A.11) we establish the inequality (A.10) for Q_n^*. Let us examine a nonsingular line of polynomial Q_n in the domain ω / S_n.

As usual, we introduce curvilinear coordinates in the vicinity of this level line. In analogy to Lemma A.3 we stratify the domain $\omega \setminus S_n$ with accuracy to a set of degree ε on the sub-domain of the form $\omega_{\rho,\rho+\Delta\rho}$. We shall examine the contour K_ρ, which surrounds the domain ω_ρ, located in $\omega \setminus S_n$. Then

$$\tau_0 \operatorname{mes} L_\rho - \tau_0 \operatorname{mes} \gamma_\rho \geqslant c \operatorname{mes} \omega_\rho$$

where γ_ρ is the part of contour K_ρ, which coincides with part γ. Then repeating the steps carried out in Lemma A.3 and noting that $\inf Q_n = 0$ on $\omega \setminus S_n$ and $\inf Q_n = 0$ on γ_n we arrive at the inequality

$$\tau_0 \int_{\omega \setminus S_n} |\nabla Q_n| \, d\omega - \tau_0 \int_{\gamma_n} Q_n \, ds \geqslant c \int_{\omega \setminus S_n} Q_n \, d\omega \qquad (A.15)$$

where γ_n is part of γ which is a piece of the boundary $\omega \setminus S_n$. Since Q_n^* coincides with Q_n in $\omega \setminus S_n$ and is equal to zero on S_n, it follows from inequality (A.15) that

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*) The symbol $W_p{}^1(\omega)$ denotes a set of functions in the domain ω which have first derivatives integrable with the degree p.

**) The notation $\operatorname{supp} h$ as usual applies to the carrier of the function h, i.e. the set of points of the plane where $h \neq 0$.

$$\tau_0 \int_\omega |\nabla Q_n^*|\, d\omega - \tau_0 \int_\gamma Q_n^*\, ds \geqslant c \int_\omega Q_n^*\, d\omega \qquad \text{(A.16)}$$

Relationship (A.10) follows from inequality (A.16) and relationships (A.14)

P r o o f o f T h e o r e m 1.1 . Let us examine an increment of functional (1.1). Then

$$\Delta J = J(u_0 + \lambda h) - J(u_0) =$$
$$= \int_\omega \left\{\lambda\mu \nabla u_0 \nabla h + \frac{\mu}{2} |\nabla h|^2 + \tau_0 |\nabla (u_0 + \lambda h)| - \tau_0 |\nabla u_0 - c\lambda h\right\} d\omega$$

Let ω_λ denote a domain where $|\nabla u_0| > \lambda^\alpha$, $\alpha < 1/2$. The increment ΔJ is written in the form

$$\Delta J = \int_\omega \left\{\lambda\mu \nabla u_0 \nabla h + \frac{\mu}{2} (\nabla h)^2\right\} d\omega + \int_\Omega \tau_0 |\lambda|\, |\nabla h|\, d\omega +$$
$$+ \int_{\omega_\lambda} \tau_0 \{|\nabla (u_0 + \lambda h)| - |\nabla u_0|\}\, d\omega + \int_\omega \tau_0 \{|\nabla (u_0 + \lambda h)| - |\nabla u_0|\}\, d\omega - \int_\omega c\lambda h\, d\omega$$
$$\omega_0 = \omega \setminus (\omega_\lambda \cup \Omega)$$

Noting that $\operatorname{mes}\{\omega \setminus (\omega_\lambda \cup \Omega)\} \to 0$ for $\lambda \to 0$ and that

$$\left|\int_\omega \{|\nabla (u_0 + \lambda h)| - |\nabla u_0|\}\, d\omega\right| \leqslant |\lambda| \int_\omega |\nabla h|\, d\omega$$

we obtain

$$\Delta J = \int_\omega \{\lambda\mu \nabla u_0 \nabla h - c\lambda h\}\, d\omega + \int_\Omega \tau_0 |\lambda|\, |\nabla h|\, d\omega +$$
$$+ \int_{\omega_\lambda} \tau_0 \{|\nabla (u_0 + \lambda h)| - |\nabla u_0|\}\, d\omega + o(\lambda) \qquad \text{(A.17)}$$

Transforming the last integral in (A.17)

$$\int_{\omega_\lambda} \{|\nabla (u_0 + \lambda h)| - |\nabla u_0|\}\, d\omega = \int_{\omega_\lambda} \frac{\nabla u_0 \nabla h}{|\nabla u_0|}\, d\omega +$$
$$+ \int_{\omega_\lambda} \frac{\lambda^2 |\nabla h|^2}{|\nabla (u_0 + \lambda h)| + |\nabla u_0|}\, d\omega + \int_{\omega_\lambda} \frac{\nabla u_0 \nabla h \{-\lambda^2 |\nabla h|^2 - 2\lambda \nabla u_0 \nabla h\}}{|\nabla u_0|\, \{|\nabla (u_0 + \lambda h)| + |\nabla u_0|\}}\, d\omega$$

We apparently obtain

$$\int_{\omega_\lambda} \frac{\lambda^2 |\nabla h|^2}{|\nabla (u_0 + \lambda h)| + |\nabla u_0|}\, d\omega = o(\lambda)$$
$$\int_{\omega_\lambda} \frac{\nabla u_0 \nabla h \{-\lambda^2 |\nabla h|^2 - 2\lambda \nabla u_0 \nabla h\}}{|\nabla u_0|\, \{|\nabla (u_0 + \lambda h)| + |\nabla u_0|\}}\, d\omega = o(\lambda)$$

Then

$$\int_{\omega_\lambda} \frac{\nabla u_0 \nabla h}{|\nabla u_0|}\, d\omega = -\int_{\omega_\lambda} \left[\operatorname{div}\left(\frac{\nabla u_0}{|\nabla u_0|}\right)\right] h\, d\omega + \int_{S_\lambda} h \left(\frac{\nabla u_0}{|\nabla u_0|}\right)\Big|_n ds$$

where S_λ is a contour surrounding ω_λ, $(\nabla u_0 / |\nabla u_0|)|_n$ is the projection of the vector on the direction of the external normal to S_λ. In this manner

$$\triangle J = -\int\limits_{\omega_\lambda}\left\{\mu\triangle u_0+\tau_0 \operatorname{div}\frac{\nabla u_0}{|\nabla u_0|}+c\right\}\lambda h\, d\omega-\int\limits_{\Omega} c\lambda h\, d\omega+$$
$$+\int\limits_{\Omega}\tau_0|\nabla\lambda h|\, d\omega+\int\limits_{S_\lambda}\frac{\nabla u_0}{|\nabla u_0|}\Big|_n ds+o(\lambda)$$

We note that if point S_λ for $\lambda \to 0$ approaches a point on the boundary a_1 of the nucleus of flow A_1, then $(\nabla u_0/|\nabla u_0|)|_n \to 1$; if however the point S_λ for $\lambda \to 0$ approaches a point on the boundary b_1 of the stagnant zone B_1, then $(\nabla u_0/|\nabla u_0|)|_n \to -1$. Consequently,

$$\int\limits_{S_\lambda}\frac{h\nabla u_0}{|\nabla u_0|}\Big|_n ds-\Big[\int\limits_{\bigcup_1^s a_i} h\, ds-\int\limits_{\bigcup_1^p b_i} h\, ds\Big]\to 0 \quad \text{for } \lambda\to 0$$

Thus,

$$\nabla J = -\int\limits_{\omega_\lambda}\left\{\mu\triangle u_0+\operatorname{div}\frac{\nabla u_0}{|\nabla u_0|}+c\right\}\lambda h\, d\omega-\int c\lambda h\, d\omega-\int c\lambda h\, d\omega$$

$$\sum_{i=1}^{s}\Big[\tau_0\int\limits_{A_i}|\nabla\lambda h|\, d\omega+\tau_0\int\limits_{a_i}\lambda h\, ds\Big]+\sum_{i=1}^{p}\Big[\tau_0\int\limits_{B_i}|\nabla\lambda h|\, d\omega+\int\limits_{b_i}\lambda h\, ds\Big]+o(\lambda) \quad (A.18)$$

We shall prove the necessity of conditions 1, 2 and 3 of criterion. Let us take h, concentrated in ω_λ; then

$$\nabla J = -\int\limits_{\omega_\lambda}\left\{\mu\triangle u_0+\operatorname{div}\frac{\nabla u_0}{|\nabla u_0|}+c\right\}\lambda h\, d\omega+o(\lambda)\geqslant 0 \quad \textbf{(A.19)}$$

From (A.19) it follows that

$$\mu\nabla u_0+\operatorname{div}[\nabla u_0/|\nabla u_0|]+c=0 \text{ в } \omega_\lambda \quad (A.20)$$

Since λ in (A.20) is arbitrary, the necessity of 1 is proven. Consequently,

$$\nabla J=\sum_1^s\Big[\tau_0\int\limits_{A_i}|\nabla\lambda h|\, d\omega+\tau_0\int\limits_{a_i}\lambda h\, ds-c\int\limits_{A_i}\lambda h\, d\omega\Big]+$$
$$+\sum_1^p\Big[\tau_0\int\limits_{B_i}|\nabla\lambda h|\, d\omega-\tau_0\int\limits_{b_i}\lambda h\, ds-c\int\limits_{B_i}\lambda h\, d\omega\Big]+o(\lambda) \quad (A.21)$$

From (A.21) we have

$$\tau_0\int\limits_{A_i}|\nabla\lambda h|\, d\omega+\tau_0\int\limits_{a_i}\lambda h\, ds-c\int\limits_{A_i}\lambda h\, d\omega\geqslant 0 \qquad (i=1,\dots,s)$$
$$\tau_0\int\limits_{B_i}|\triangle\lambda h|\, d\omega-\tau_0\int\limits_{b_i}\lambda h\, ds-c\int\limits_{B_i}\lambda h\, d\omega\geqslant 0 \qquad (i=1,\dots,p) \quad (A.22)$$

Lemmas A.3 and A.4 confirm that conditions 2 and 3 of criterion result from inequality (A.22). The necessity of conditions is proven.

S u f f i c i e n c y . Let conditions 1, 2 and 3 of criterion be fulfilled. Then we have from Lemmas A.3 and A.4 and the representation of the reansformation of functional (A.18)

$$J(u_0+\lambda h)-J(u_0)+o(\lambda)\geqslant 0$$

From this it follows that u_0 is either a critical point of functional (1.1) or it produces a weak minimum. From Lemmas A.1 and A.2 it follows that u_0 in these cases gives an absolute minimum of functional (1.1). The criterion has been proven.

P r o o f o f L e m m a 1.1 . Functional $M(K)$ is bounded from below because $\inf M(K) \geq -\operatorname{mes} D$. By virtue of compactness of a set of curves with limited length there exists a contour K' for which $\inf N(K) = N(K')$. Evidently contour K' is convex at internal points D. Let us examine three sufficiently closely situated internal points M_1', M_2' and M_3' of region D. These points lie on contour K' (Fig.15). It is further assumed that the arc $M_1' M_2' M_3'$ of contour K' consists of internal points D. In this manner the segment $[M_1', M_3']$ is contained in the domain K'^*, bounded by contour K'. Let K' designate a domain enclosed between segment $[M_1', M_3']$ and the arc $M_1'M_2'M_3'$. We shall also examine an arbitrary convex arc $M_1'M_2''M_3'$ located in D. Let K_1'' designate a sub-domain D, enclosed between the arc $M_1'M_2''M_3'$ and the segment $[M_1', M_3']$. Then it is easy to see that

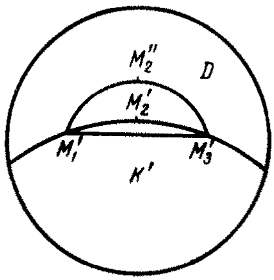

Fig. 15

$$\tau_0 \operatorname{mes}(M_1'M_2''M_3') - c \operatorname{mes} K_1'' \geqslant \tau_0 \operatorname{mes}(M_1'M_2'M_3') - c \operatorname{mes} K_1'$$

Thus, if a new system of coordinates is introduced orienting the axis O_x along the segment $[M_1', M_3']$, the axis O_x perpendicular and locating the origin in point M_1' then the arc M_1', M_2', M_3' minimizes the integral

$$\int_0^{\chi} (\tau_0 \sqrt{1 + y'^2} - cy)\, dx \qquad (\chi = \operatorname{mes}[M_1'M_3'1])$$

for conditions $y(0) = 0$; $y(\operatorname{mes}[M_1', M_3']) = 0$. It is easy to verify that extremals of functional (A.23) are peripheries with a radius τ_0/c. Confirmation of contact between K' and d can be obtained directly utilizing the well-known theorem on one-sided variations [6]. Lemma 1.1 is proven.

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